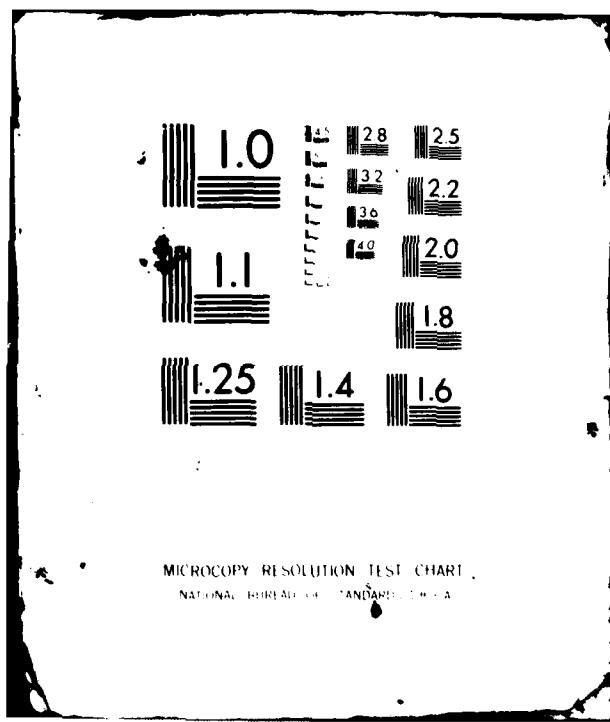


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A SEMIGROUP APPROACH TO EQUATIONS WITH  
INFINITE DELAY AND APPLICATION TO A  
PROBLEM OF VISCOELASTICITY

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A SEMIGROUP APPROACH TO EQUATIONS WITH INFINITE DELAY  
AND APPLICATION TO A PROBLEM OF VISCOELASTICITY

M. Renardy

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ABSTRACT

A semigroup approach to differential-delay equations is developed which seems more suitable for certain partial integro-differential equations than the standard theory. The method is applied to prove a local-time existence theorem for equations of the form  $u_{tt} = q(u_{xt}, u_x^t, t)_x$ , where  $\frac{\partial q}{\partial u_{xt}} > 0$ .

On a formal level, it is demonstrated that the stretching of filaments of viscoelastic liquids can be described by an equation of this form.

AMS(MOS) Subject Classifications: 34K05, 35K55, 45K05, 47D05, 76A10

Key Words: Quasilinear Parabolic Equations, Functional Differential Equations, Semigroup Theory, Viscoelastic Liquids

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## SIGNIFICANCE AND EXPLANATION

An existence theorem for differential equations of the form  $u_{tt} = q(u_{xt}, u_x^t, t)_x$  with  $\frac{\partial g}{\partial u_{xt}} > 0$  is proved. Here  $u_x^t$  denotes the "history" of  $u$  up to time  $t: u_x^t(s) = u_x(t-s)$ . Equations of this nature arise in the theory of viscoelastic liquids. Specifically, we consider the problem of stretching thin filaments. The constitutive law is assumed to be such that the stress consists of the following three parts: a pressure term, a Newtonian contribution and a "genuine" polymeric part, which is a smooth functional of the strain history. In a one dimensional approximation this leads to an equation of the described form.

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A SEMIGROUP APPROACH TO EQUATIONS WITH INFINITE DELAY  
AND APPLICATION TO A PROBLEM IN VISCOELASTICITY

M. RENARDY

1. Introduction.

Consider a linear functional differential equation of the form

$$(1.1) \quad y_t = Ly^t,$$

where  $y^t$  denotes the history of  $y$  on a certain interval  $[t-r, t]$ :  $y^t(s) = y(t-s)$ , and  $L$  is a continuous linear operator from  $C[-r, 0]$  into  $\mathbb{R}$ .

Equations of this kind can be regarded as evolution problems on  $C[-r, 0]$  and can be studied using semigroup theory. A generator  $A$  of a  $C^0$ -semigroup can be associated to (1.1) as follows (see [2]):

$$(1.2) \quad A\varphi(s) = \begin{cases} \frac{d\varphi(s)}{ds} & \text{for } -r < s < 0 \\ L\varphi & \text{for } s = 0. \end{cases}$$

This approach has been very successfully applied to systems of ordinary differential equations with finite delays. If, however, partial differential equations and infinite delays are considered, the operator  $A$  has some inconvenient properties. For instance, the semigroup generated by  $A$  is not analytic even for  $L = 0$ . Moreover, changes may occur in the spectrum of  $A$  for  $r \rightarrow \infty$ . Let again  $L = 0$ . Then, for every finite  $r$ , the spectrum of  $A$  consists only of the simple eigenvalue zero. If, however,  $A$  is considered an operator on  $\lim_{b \rightarrow \infty} C_b^{(-\infty, 0]}$  (the space of bounded continuous functions that have a limit at  $-\infty$ ), its spectrum is the whole left half plane. Such properties are disturbing, if one studies partial integro-differential equations such as the following

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$$(1.3) \quad u_t(x,t) = u_{xx}(x,t) + \int_{-\infty}^t a(t-s)f(u(x,s),u_x(x,s))ds$$

with, e.g., Dirichlet or Neumann boundary conditions. One is tempted to regard (1.3) as a perturbation of the heat equation and apply the results of Sobolevskii [10] to prove existence theorems. If, however, one follows the recipe outlined above, then even the heat equation itself is no longer associated to an analytic semigroup.

For these reasons, we choose an alternative method of representing equations with infinite delays as an abstract evolution problem. Let  $F$  be a smooth (possibly nonlinear) functional from  $C_b^{\lim}(-\infty, 0]$  into  $\mathbb{R}$ . Moreover, for every function  $\varphi \in C_b^{\lim}(-\infty, 0]$  we define  $T_r \varphi \in C_b^{\lim}(-\infty, 0]$  by  $T_r \varphi(s) = \varphi(s+r)$  for each  $r < 0$ . Then with  $F$  we associate the following operator  $\hat{F}: C_b^{\lim}(-\infty, 0] \rightarrow C_b^{\lim}(-\infty, 0]$ :

$$(1.4) \quad \hat{F}(\varphi)(s) = F(T_s \varphi).$$

There is some temptation to rewrite the equation  $y_t = F(y^t)$  in the abstract form  $\hat{y}_t = \hat{F}(\hat{y})$ ,  $\hat{y} \in C_b^{\lim}(-\infty, 0]$ . However, this is incorrect. The reason is that functional differential equations of this kind are usually associated with an initial condition

$$(1.5) \quad y(t) = y_0(t) \text{ for } t < 0,$$

and  $y_0$  is not required to satisfy the equation. We can, however, define  $f(t)$  by

$$(1.6) \quad f(t) = \begin{cases} \frac{\partial y_0(t)}{\partial t} - F(y_0^t) & \text{for } t < 0 \\ 0 & \text{for } t > 0. \end{cases}$$

It will be assumed that  $y_0$  is such that  $f$  is continuous (including the point  $t = 0$ ), bounded and such that  $\lim_{t \rightarrow -\infty} f(t)$  exists. Then  $\hat{f}(t) \in C_b^{\lim}(-\infty, 0]$  can be defined by

$$(1.7) \quad \hat{f}(t)(s) = f(t+s).$$

This permits to rewrite the equation  $y_t = F(y^t)$  in the form

$$(1.8) \quad \hat{y}_t = \hat{F}(\hat{y}) + \hat{f}.$$

In this paper this approach will be applied to equations of the form

$$(1.9) \quad \begin{aligned} u_{tt} &= g(u_{xt}, u_x^t)_x, \quad x \in [-1, 1] \\ g(u_{xt}, u_x^t) &= f_{\pm 1}(t) \quad \text{at } x = \pm 1. \end{aligned}$$

It will be shown that, if  $\frac{\partial g}{\partial u_{xt}} > 0$ , the Sobolevskii theory of quasilinear parabolic equations can be applied to this problem to prove an existence theorem locally in time.

In §3, we will show how the problem of stretching thin filaments of polymeric liquids reduces to an equation of the type (1.9). For this, we use a formal approximation of a similar nature as those used in rod theory [7]. Other problems that our result would apply to are the exact one-dimensional motions considered by Coleman, Gurtin and Herrera [1].

Remark.

A little bit of care is needed when proceeding to (1.8). Namely, we must make sure that the abstract solution  $\hat{y}(t)(s)$  actually corresponds to a function  $y(t+s)$ , in other words,  $T_s \hat{y}(t) = \hat{y}(t+s)$ . For this, observe that, if  $\hat{y}(t)$  is a solution of (1.8), then  $\hat{z}(t) = T_s \hat{y}(t)$  solves the equation

$$(1.10) \quad \hat{z}_t = \hat{F}(\hat{z}) + T_s \hat{f}.$$

So does  $\hat{z}'(t) = \hat{y}(t+s)$ . Hence, if you have a uniqueness theorem for (1.10), you are done. Such a uniqueness theorem will automatically come out of the considerations leading to our existence result.

Acknowledgement.

I thank M. E. Gurtin for his criticisms, which greatly helped to improve this paper.

2. The Main Result.

We study the following problem

$$(2.1) \quad \begin{aligned} u_{tt} &= g(u_{xt}, u_x^t) \quad \text{for } x \in [-1, 1] \\ g(u_{xt}, u_x^t) &= f_{\pm 1}(t) \quad \text{at } x = \pm 1. \end{aligned}$$

In view of the application we have in mind, we require  $g$  to be defined only on an open subset of the history space  $C_b^{\lim}(-\infty, 0]$ . We denote this open subset by  $\Omega$ . We assume then

- (i)  $g$  is a smooth function from  $\mathbb{R} \times \Omega$  into  $\mathbb{R}$ . Moreover, there exists  $\epsilon > 0$  such that  $D_1 g > \epsilon$  on  $\mathbb{R} \times \Omega$ . (By  $D_1 g$ ,  $D_2 g$  we denote the Fréchet derivatives of  $g$  w.r. to the first and second argument.).

Let us first study the boundary conditions. They yield equations of the form

$$(2.2) \quad g(y_t, y^t) = f(t)$$

(we have substituted  $y$  for  $u_x(\pm 1)$ ). According to our assumptions on  $g$ , we can resolve this equation w.r. to  $y_t$ , leading to an equation of the form

$$(2.3) \quad y_t = h(y^t, f(t))$$

where  $h$  is smooth from  $\Omega \times \mathbb{R}$  into  $\mathbb{R}$ .

As usual, we study (2.2) for  $t > 0$  with an initial condition  $y(t) = y_0(t)$  for  $t < 0$ . On this initial condition we assume

- (ii)  $y_0 \in \Omega$  has a continuous bounded derivative, and  $\lim_{s \rightarrow -\infty} \frac{dy_0(s)}{ds} = 0$ .  
Moreover, the compatibility condition

$$g\left(\frac{dy_0}{dt}(0), y_0\right) = f(0)$$

is satisfied.

We now define  $f(t)$  by

$$(2.4) \quad f(t) = \begin{cases} g\left(\frac{dy_0(t)}{dt}, T_t y_0\right) & \text{for } t < 0 \\ \text{the given } f & \text{for } t \geq 0. \end{cases}$$

As in the introduction, let  $\hat{y}(t)(s) = y(t+s)$ ,  $\hat{f}(t)(s) = f(t+s)$ . Then (2.3) is rewritten in the abstract form

$$(2.5) \quad \hat{y}_t = \hat{h}(\hat{y}, \hat{f}(t))$$

where  $\hat{h}$  is smooth from  $\Omega \times C_b^{\lim}(-\infty, 0]$  into  $C_b^{\lim}(-\infty, 0]$ . The Picard-Lindelöf theorem now immediately yields the following result.

Theorem 2.1.

Let (i), (ii) hold. Then, for some  $T > 0$ , equation (2.5) with the initial condition  $\hat{y}(0) = y_0$  has a unique solution  $\hat{y} \in C^1([0, T], C_b^{\lim}(-\infty, 0])$ .

We now turn to the study of (2.1). It is convenient to rewrite this equation as a system. We put  $u_x = p$ ,  $u_t = \tilde{r}$ ,  $u_{xx} = q$ . Then (2.1) is equivalent to

$$(2.6) \quad \begin{aligned} p_t &= \tilde{r}_x \\ q_t &= \tilde{r}_{xx} \\ \tilde{r}_t &= D_1 q(\tilde{r}_x, p^t) + \tilde{r}_{xx} + D_2 g(\tilde{r}_x, p^t) + q^t. \end{aligned}$$

Remark.

We have assumed here that the pointwise derivative  $q = p_x$  is actually the derivative in the sense to the space  $C_b^{\lim}(-\infty, 0]$ . This is justified if  $q(t+s, x)$  is uniformly bounded and uniformly continuous for  $x \in [-1, 1]$ ,  $s \in (-\infty, 0]$ . Our assumptions on the initial history will be such that this is implied.

From our study of the boundary problem, we know solutions  $u_x(t,1) = y_1(t)$ ,  $u_x(t,-1) = y_{-1}(t)$ . With (2.6) we must then associate the boundary conditions

$$(2.7) \quad \tilde{r}_x(t,1) = \frac{dy_1(t)}{\partial t}, \quad \tilde{r}_x(t,-1) = \frac{dy_{-1}(t)}{\partial t}.$$

Letting  $r = \tilde{r} - \frac{1}{2}x\left(\frac{dy_1(t)}{\partial t} + \frac{dy_{-1}(t)}{\partial t}\right) - \frac{1}{4}x^2\left(\frac{dy_1(t)}{\partial t} - \frac{dy_{-1}(t)}{\partial t}\right)$ ,

we obtain homogeneous Neumann conditions for  $r$ .

Before we reformulate (2.6) as an abstract evolution problem, we need to introduce some function spaces. Let  $H^n$  denote the space of all real-valued functions on  $[-1,1]$ , which have  $n$  square integrable derivatives. By  $C_b^{\lim}((-\infty,0], H^n)$  we denote the space of continuous, bounded functions from  $(-\infty,0]$  into  $H^n$ , which converge to a limit at  $-\infty$ . We assume

(iii)  $q$  (regarded as acting pointwise w.r. to the space variable) is a smooth mapping from  $\mathbb{R} \times \tilde{\Omega}$  into  $H^1$ , where  $\tilde{\Omega}$  is an open subset of  $C_b^{\lim}((-\infty,0], H^1)$ . On  $\mathbb{R} \times \tilde{\Omega}$ , we have  $D_1 q > \epsilon > 0$ .

We are studying an initial history value problem, i.e.  $u$  is given for

$t < 0$ . We assume that this initial history is such that

(iv)  $u_t$  lies in  $C_b^{\lim}((-\infty,0], H^3)$ , and  $u_{tt}$  lies in  $C_b^{\lim}((-\infty,0], H^1)$ . Moreover,  $u_x$  lies in  $C_b^{\lim}((-\infty,0], H^2 \cap \tilde{\Omega})$ .

The history of  $u$  provides histories for  $p, q$  and  $r$  in the obvious way.

We make sure that the history satisfies (2.6) by adding an inhomogeneous term  $f^*(t,x)$  to the right side of the third equation. We assume a compatibility condition

(v) The initial history is such that  $f^*(0,x) = 0$ .

Finally, we assume the following smoothness conditions

(vi)  $f^*$  (regarded as valued in  $H^1$ ) is uniformly Hölder continuous in  $t$ . Moreover,  $y_1$  and  $y_{-1}$  (the boundary values of  $u_x$ ) are

in  $C^2$ , and  $\frac{d^2y_1}{dt^2}, \frac{d^2y_{-1}}{dt^2}$  are uniformly Hölder continuous.

For abbreviation, let us put

$$\varphi(t, x) = \frac{1}{2}x\left(\frac{dy_1(t)}{dt} + \frac{dy_{-1}(t)}{dt}\right) - \frac{1}{4}x^2\left(\frac{dy_1(t)}{dt} - \frac{dy_{-1}(t)}{dt}\right),$$

$$\varphi_1(t, x) = \frac{\partial \varphi(t, x)}{\partial x}, \quad \varphi_2(t, x) = \frac{\partial^2 \varphi(t, x)}{\partial x^2}, \quad \varphi_3(t, x) = \frac{\partial \varphi(t, x)}{\partial t}.$$

As before, we define  $\hat{p}(t)(s) = p(t-s)$  etc. Equation (2.6) now assumes the following form

$$(2.8) \quad \begin{aligned} \hat{p}_t &= \hat{r}_x + \hat{\varphi}_1 \\ \hat{q}_t &= \hat{r}_{xx} + \hat{\varphi}_2 \\ \hat{r}_t &= D_1 \hat{g}(\hat{r}_x + \hat{\varphi}_1, \hat{p})(\hat{r}_{xx} + \hat{\varphi}_2) + D_2 \hat{g}(\hat{r}_x + \hat{\varphi}_1, \hat{p})\hat{q} - \hat{\varphi}_3 + \hat{f}^* \end{aligned}$$

with the boundary condition  $\hat{r}_x = 0$  at  $x = \pm 1$ .

We quote the following result of Sobolevskii [10]

### Theorem 2.2.

Consider an evolution equation

$$(2.9) \quad v_t + A(t, v)v = f(t, v)$$

in a Banach space  $E$ . Assume the following are true:

$A_0 = A(0, v_0)$  is a linear operator whose domain  $D$  is dense in  $E$ .

Let there be  $d \in \mathbb{R}$  such that, for  $\operatorname{Re} \lambda > d$ , the operator  $A_0 + \lambda I$  has a bounded inverse, and

$$(2.10) \quad \|(\mathbf{A}_0 + \lambda \mathbf{I})^{-1}\| \leq C(|\lambda| + 1)^{-1}.$$

For some  $\alpha \in (0,1)$  and for any  $v \in E$ ,  $\|v\| \leq R$ , let the operator

$A(t, A_0^{-\alpha} v)$  be defined in  $D$  and

$$(2.11) \quad \|[\mathbf{A}(t, \mathbf{A}_0^{-\alpha} v) - \mathbf{A}(\tau, \mathbf{A}_0^{-\alpha} w)]\mathbf{A}_0^{-1}\| \leq C(R)(|t-\tau|^\epsilon + \|v-w\|), \quad \epsilon \in (0,1)$$

for any  $0 \leq t, \tau \leq T$ ,  $\|v\| \leq R$ ,  $\|w\| \leq R$ . In this region, let

$$(2.12) \quad \|f(t, \mathbf{A}_0^{-\alpha} v) - f(\tau, \mathbf{A}_0^{-\alpha} w)\| \leq C(R)(|t-\tau|^\epsilon + \|v-w\|).$$

Finally, for some  $\beta > \alpha$ , let  $v_0 \in D(\mathbf{A}_0^\beta)$ , and let  $\|\mathbf{A}_0^\alpha v_0\| \leq R$ . Then there exists one and only one solution of equation (2.9) which is defined on an interval  $[0, t_0]$ , is continuous for  $t \in [0, t_0]$ , is continuously differentiable for  $t > 0$ , and satisfies the initial condition

$$(2.13) \quad v(0) = v_0.$$

We apply this to (2.8) by making the following identifications. Let  $E$  denote the space  $C_b^{\lim}((-\infty, 0], H^1)^3$  and let  $v = (\hat{p}-\hat{p}_0, \hat{q}-\hat{q}_0, \hat{r}-\hat{r}_0)$ , where  $\hat{p}_0, \hat{q}_0, \hat{r}_0$  denote the given initial conditions. The operator  $A(t, v)$  is defined by  $-A(t, v)(\hat{p}', \hat{q}', \hat{r}') = (\hat{r}'_x, \hat{r}'_{xx}, D_1 \hat{q}(\hat{r}_x + \hat{p}_1, \hat{p}) \hat{r}'_{xx})$  incorporating the boundary condition  $\hat{r}'_x = 0$  and  $f(t, v)$  is identified with the remainder of the right hand side of (2.8). Then the initial condition  $v_0 = 0$  is in  $D(\mathbf{A}_0)$ , i.e. we can choose  $\beta = 1$ . Moreover, for appropriate choice of  $R$ , it is clear that (2.11), (2.12) hold with  $\alpha = 1/2$ , and  $\epsilon$  the Hölder exponent of condition (vi). Condition (2.10) finally follows from the positivity of  $D_1 \hat{q}$  and the fact that, for positive  $\varphi(x)$ , the operator  $r \mapsto \varphi(x)r_{xx}$  with homogeneous Neumann conditions is selfadjoint in  $H^1$ , provided that  $H^1$  is equipped with the scalar product  $((p, q)) =$

$\int_{-1}^1 \frac{1}{\varphi(x)} p(x)q(x)dx + \int_{-1}^1 p_x(x)q_x(x)dx$ . This product is equivalent to the usual one.

We thus obtain

Theorem 2.3.

Let hypotheses (i)-(vi) hold. Then, for some  $T > 0$ , equation (2.8) has a unique solution in  $C^1([0,T], C_b^{\lim}(-\infty, 0], H^1)^3)$ , which satisfies the given initial conditions.

Remarks.

Certain models for viscoelastic liquids express the stress as a functional not only of the history of strain, but also of the history of the rate of strain (see e.g. [11]). In this case we will have to deal with equations of the form

$$(2.4) \quad u_{tt} = g(u_{xt}, u_x^t, u_{xt}^t)x.$$

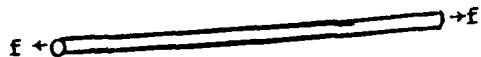
If  $D_3q$  is small compared to  $D_1q$ , it can be shown by perturbation arguments that the assumptions necessary for the theorems above remain still valid.

In many cases one has to deal with functionals that, although they involve the rate of strain, also have a smoothing property. The stress is (at least) as often differentiable w.r. to time as the strain, and no derivative is lost.

E.g. take a constitutive law of the form  $\sigma(t) = \int_{-\infty}^t a(t-s)f(u_{xt}(s))ds$ . This leads to  $\sigma_t(t) = \int_{-\infty}^t a'(t-s)f(u_{xt}(s))ds + a(0)f(u_{xt}(t))$ . Thus  $\sigma_t$  is given by an expression involving only  $u_{xt}$  and not  $u_{xtt}$ . If the history of  $u_{xt}$  enters (2.4) via such expressions, it is possible to apply the same methods as before to the time differentiated version of (2.4). The terms involving histories are then still of "lower order" and can be included in the "f-part". We leave the details to the reader.

### 3. The Stretching of Thin Filaments of Viscoelastic Liquids.

We consider the problem of stretching a thin filament of a viscoelastic liquid by a force  $f$  applied to its ends:



We shall derive an equation describing this problem in an analogous fashion as I did in [8] assuming a special constitutive law. We start with the general three-dimensional equations of continuum mechanics and then introduce a one-dimensional approximation analogous to rod theory [7].

Let  $(\xi^1, \xi^2, \xi^3)$  denote "body coordinates", i.e. coordinates labelling a specific particle in the fluid. These coordinates can be identified with the position of the particle in space, when the fluid is in a certain "reference state". This reference state will be chosen such that the filament is cylindrical in this state. By  $(y^1, y^2, y^3)$  we denote Cartesian coordinates in space. We are interested in finding trajectories of particles, i.e. a functional dependence  $y: (\xi^1, \xi^2, \xi^3, t)$ .

The deformation gradient  $F$  is defined by  $F^i_j = \frac{\partial y^i}{\partial \xi^j}$  and the right Cauchy-Green strain tensor is defined by

$$(3.1) \quad C = F^T F.$$

Following Lodge [3], we introduce a Lagrangian stress tensor  $S$ , which is related to the Cauchy stress tensor  $T$  by

$$(3.2) \quad S = F^{-1} T (F^T)^{-1}.$$

$S$  is related to  $C$  by a constitutive law. We assume that this constitutive equation consists of three terms: a pressure term, a Newtonian viscosity and a "genuine" polymeric contribution, which is a smooth functional of the strain history. I.e., we have

$$(3.3) \quad S = -p C^{-1} - n \frac{\partial}{\partial t} C^{-1} + F(C^t).$$

The incompressibility condition is expressed by

$$(3.4) \quad \det C = 1.$$

The equation of motion is then given by

$$(3.5) \quad \rho y_{tt}^k = \text{Div}(FS)$$

where Div refers to the body coordinate system, or, in coordinates

$$(3.6) \quad \rho y_{tt}^k = \sum_{r,s} \frac{\partial y^k}{\partial \zeta^r} \frac{\partial}{\partial \zeta^s} S^{rs} + \frac{\partial^2 y^k}{\partial \zeta^r \partial \zeta^s} S^{rs}.$$

We have to supplement (3.6) by the boundary conditions. Let  $f$  denote the force acting on a surface divided by the surface area in the reference state.  $f$  is referred to space coordinates. Then the boundary condition reads

$$(3.7) \quad F \cdot S \cdot n = f,$$

where  $n$  denotes the unit normal of the surface in the body coordinate system.

We now specialize our considerations to the filament. Coordinates are chosen such that  $\zeta^1$  and  $y^1$  are in the direction of the filament, and the other coordinates are transverse. It is assumed that in the reference state the filament is axisymmetric and cylindrical, i.e.  $\zeta^1$  ranges over a finite interval, say  $[-1,1]$ , and  $r = \sqrt{(\zeta^2)^2 + (\zeta^3)^2}$  ranges from 0 to  $\delta$ , the radius of the filament. We shall assume that  $\delta$  is small. In a formal argument, we shall derive an equation for the terms of highest order w.r. to  $\delta$ . The argument is analogous to the one given by Narihani [7] for the problem of longitudinal elastic waves in a thin rod.

We assume that the solutions  $y^1, y^2, y^3$  and the pressure  $p$  are such that they depend smoothly on  $r$ , and that the coefficients in a Taylor expansion w.r. to powers of  $r$  are smooth functions of  $\delta$ . Let us put  $\zeta^2 = \frac{\zeta^2}{\delta}$ ,  $\zeta^3 = \frac{\zeta^3}{\delta}$ ,  $\hat{r} = \frac{r}{\delta}$ , so that the lateral surface corresponds to

$\hat{r}=1$ . For the solutions we now make the Ansatz:

$$(3.8) \quad \begin{aligned} y^1(\zeta^1, \delta\zeta^2, \delta\zeta^3) &= \sum_{v=0}^N \delta^{2v} P_v(\zeta^1, \hat{r}^2) + o(\delta^{2N+2}) \\ y^2(\zeta^1, \delta\zeta^2, \delta\zeta^3) &= \delta\zeta^2 \sum_{v=0}^N \delta^{2v} Q_v(\zeta^1, \hat{r}^2) + o(\delta^{2N+3}) \\ y^3(\zeta^1, \delta\zeta^2, \delta\zeta^3) &= \delta\zeta^3 \sum_{v=0}^N \delta^{2v} R_v(\zeta^1, \hat{r}^2) + o(\delta^{2N+3}) \end{aligned}$$

$$p(\zeta^1, \delta\zeta^2, \delta\zeta^3) = \sum_{v=0}^N \delta^{2v} R_v(\zeta^1, \hat{r}^2) + o(\delta^{2N+2})$$

where  $P_v, Q_v, R_v$  are polynomials of  $v$ th degree in  $\hat{r}^2$ . We insert this Ansatz into (3.4), (3.6) and the boundary conditions (3.7) on the lateral surface of the filament. By retaining only lowest order terms in  $\delta$ , we shall obtain an equation for  $P_0(\zeta^1)$ . In the following analysis we shall carry out the series expansions (3.8) only so far as is necessary for this purpose.

For the Cauchy-Green tensor we find

$$(3.9) \quad C = \begin{pmatrix} \left( \frac{\partial P_0}{\partial \zeta^1} \right)^2 & \delta\zeta^2 \cdot \varphi & \delta\zeta^3 \cdot \varphi \\ \delta\zeta^2 \cdot \varphi & Q_0^2 & 0 \\ \delta\zeta^3 \cdot \varphi & 0 & Q_0^2 \end{pmatrix} + o(\delta^2)$$

$$\text{where } \varphi = \frac{\partial P_0}{\partial \zeta^1} \cdot \frac{\partial P_2}{\partial \hat{r}} \hat{r}^{-1} + \frac{\partial Q_0}{\partial \zeta^1} Q_0.$$

Thus, in the lowest order, the incompressibility condition (3.4) yields

$$(3.10) \quad \left( \frac{\partial P_0}{\partial \zeta^1} \right)^2 Q_0^4 = 1.$$

Next, let us consider the boundary conditions on the lateral surface. At a boundary point where  $\zeta^3 = 0$  these yield the equations  $S^{21} = S^{22} = S^{23} = 0$ .

(Because of the radial symmetry it suffices to consider these boundary points; if the traction vanishes there, it does so everywhere else).  $s^{23}$  vanishes identically as a result of the radial symmetry. For  $s^{22}$  we obtain the following terms of order  $O(1)$ :

$$(3.11) \quad s^{22} = -R_0 \zeta_0^{-2} - n \frac{\partial}{\partial t} (\zeta_0^{-2}) + F^{22}(C_0^t)$$

where  $C_0$  denotes the zeroth order term in  $C$ . Moreover, we have to satisfy

$$(3.12) \quad s^{21} = 0.$$

It follows from the principle of material frame indifference that, if  $C$  is diagonal, then  $S$  must be diagonal, too (see e.g. [3]). Hence (3.12) is satisfied in  $O(\delta)$ , if  $\varphi = 0$ . The equation of motion now yields the following equation for  $p_0$ :

$$(3.13) \quad \rho \frac{\partial^2 p_0}{\partial t^2} = \frac{\partial p_0}{\partial \zeta^1} \left\{ \frac{\partial}{\partial \zeta^1} \left( -R_0 \left( \frac{\partial p_0}{\partial \zeta^1} \right)^{-2} - n \frac{\partial}{\partial t} \left( \frac{\partial p_0}{\partial \zeta^1} \right)^{-2} + F^{11}(C_0^t) \right) \right\} \\ + \frac{\partial^2 p_0}{(\partial \zeta^1)^2} \cdot \left( -R_0 \left( \frac{\partial p_0}{\partial \zeta^1} \right)^{-2} - n \left( \frac{\partial p_0}{\partial \zeta^1} \right)^{-2} + F^{11}(C_0^t) \right).$$

In order to simplify notation, let us write  $u$  for  $p_0$  and  $x$  for  $\zeta^1$ .

Since  $C_0$  depends only on  $u_x$ , we shall also write  $F^{11}(C_0^t) = F^1(u_x^t)$ ,

$F^{22}(C_0^t) = F^2(u_x^t)$ . When we substitute  $R_0$  from (3.11), we obtain the

following equation from (3.13)

$$(3.14) \quad \rho u_{tt} = 3n \frac{\partial^2}{\partial x \partial t} \left( -\frac{1}{u_x^2} \right) + \frac{\partial}{\partial x} \left\{ u_x F^1(u_x^t) - u_x^{-2} F^2(u_x^t) \right\}.$$

Finally, we have to derive the boundary conditions at the ends of the filament. As noted in [7], expansions of the type (3.8) can not be used to satisfy these in every order in  $\delta$ , and a "boundary layer" arises. However, for the first order approximation, we need not deal with this. When we insert (3.8) into the boundary conditions at the ends, then all traction components not in the direction of the filament are  $O(\delta)$ . For the longitudinal

traction, we find in order  $O(1)$ :

$$(3.15) \quad 3n \frac{\partial}{\partial t} \left( -\frac{1}{u_x^2} \right) + \{ u_x F^1(u_x^t) - u_x^{-2} F^2(u_x^t) \} = f(t)$$

where  $f(t)$  denotes the force acting on the end, divided by the cross-sectional area in the reference state,  $\pi\delta^2$ .

Remark.

Our analysis in chapter 2 breaks down if  $n = 0$ . However, we have seen from numerical studies [5], [6] of a specific model that, for certain fluids (like molten polyethylene),  $n$  must be assumed very large in order to have significant influence on the solution. Experimental values for  $n$  are not available. Thus a theory that includes the case  $n = 0$  seems to have more than academic interest. The equations are not parabolic in this case, but can for a wide class of constitutive laws be classified as hyperbolic. This will be discussed in a forthcoming paper [9]. In a restricted situation, Lodge, McLeod and Nohel [4] have discussed solutions to the boundary equation (3.15) in the limit  $n \rightarrow 0$ .

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